Existence of solution for a p(x)-Kirchhoff type with singular weights

A. Zaki[1], A. Hamydy[2], N. Tsouli[3]

[2]Centre Regional des Metiers de l'Education et de la Formation, Tanger, Morocco.
[1,3]Laboratory Nonlinear Analysis, Department of Mathematics, Faculty of Science, University Mohammed I, Oujda, Morocco.

E-mail: a.zaki@ump.ac.ma[1], a.hamydy@yahoo.fr[2], tsouli@hotmail.com[3]

Abstract

In this paper we will use the compact embedding $W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$ and Galerkin′s approach to prove the existence of at least one solution to a p(x)-Kirchhoff problem with convection term and singular weights.

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1 Introduction

We consider the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{d^\alpha}|\nabla u|^{p(x)}\mathrm{dx}\right)\ \mathrm{div}\left(\frac{1}{d^\alpha}|\nabla u|^{p(x)-2}\nabla u\right) &= \frac{1}{d^\beta}h(x,u)+\frac{1}{d^\gamma}g(x,\nabla u)\ \text{ in } \Omega \\ u &= 0, \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta_{p(x)}$ is the standard p(x)-Laplacian operator and $p, q \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}), h^- > 1\}$ where $h^- = \min_{x\in\overline{\Omega}} h(x), h^+ = \max_{x\in\overline{\Omega}} h(x)$. Assume that M, h, and g satisfy the following assumptions:

(H_1) $\quad M : (0, +\infty) \longrightarrow (0, +\infty)$ continuous and $m_0 = inf_{s>0}M(s) > 0.$

(H_2) $h : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is locally Holder continuous, there exist the positive constants $a_1 \in L^{p'(x)}(\Omega, \frac{1}{d^\beta})$, $b_1 \in L^{\frac{p(x)}{p(x)-(r_1(x)+1)}}(\Omega, \frac{1}{d^\beta})$ such that

$$h(x,t) \le a_1 + b_1|t|^{r_1} \qquad \forall (x,t) \in \Omega \times \mathbb{R}$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ and $0 < r_1^- \le r_1(x) \le r_1^+ < p^- - 1$.

(H_3) $g : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally Holder continuous function and there exist positive constants $a_2 \in L^{p'(x)}(\Omega, \frac{1}{d^\gamma})$, $b_2 \in L^{\frac{p(x)}{p(x)-r_2(x)}}(\Omega, \frac{1}{d^\gamma}) \cap L^\infty(\Omega)$ such that:

$$g(x,\delta) \le a_2 + b_2|\delta|^{r_2(x)} \qquad \forall (x,\delta) \in \Omega \times \mathbb{R}^n,$$

where $0 < r_2^- \le r_2(x) \le r_2^+ < p^- - 1$.

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Note that several authors have extensively studied the semi-linear case ($p(x) = p = 2$). We quote for example [12, 13, 14, 15, 16]. Later, many researches are interested in these problems with $p > 2$, we refer the reader to [17, 18, 19, 20, 21, 22]. In [22], the author considered the problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \mathrm{dx}\right) \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f(x,u,\nabla u) \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

he prove the existence of the positive solution of the problem (1.2).
Inspired by the papers mentioned above and the reference therein, we shall generaloze the compact and continues embedding introduced by Pavel Drbek and Jess Hernndez (cf. [1]). Futhermore, as applications we use the Galirkin method to prove that the problem (1.1) admits at least one nontrivial weak solution.
Now, we will introduce our main results.

Theorem 1.1. Suppose $(H_1) - (H_3)$ and $0 \leq \beta, \gamma < \alpha + p^-$. Then, the problem (1.1) admits at least one nontrivial weak solution.

2 Preliminaries

We denote by $\left(L^{p(\cdot)}(\Omega, \vartheta(x)), \|\cdot\|_{p(x)}\right)$ and $\left(W^{1,p(\cdot)}(\Omega, \vartheta(x)), \|.\|_{1,p(x)}\right)$ the usual the weighted variable exponent Lebesgue space and Sobolev space, respectively.
For $p(.) \in L^\infty_+(\Omega)$ and $\vartheta \in L^1_{Loc}(\Omega)$ such that $\vartheta > 0$ almost everywhere in Ω. We consider the weighted variable exponent Lebesgue space by:

$$L^{p(x)}(\Omega, \vartheta) = \left\{u: \quad \Omega \longrightarrow \mathbb{R}, \quad \text{measurable and} \quad \int_\Omega \vartheta(x)|u(x)|^{p(x)}dx < \infty\right\},$$

with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega,\vartheta)} = \|u\|_{p(x)} = \inf\left\{\mu > 0/ \ \rho_{p(.),\vartheta}(\frac{u}{\mu}) = \int_\Omega \vartheta(x)\left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

and the space $(L^{p(x)}(\Omega, \vartheta), \|.\|_{p(x)})$ is a Banach.
Moreover, $u \in L^{p(x)}(\Omega, \vartheta)$ if and only if $\|u\|_{L^{p(x)}(\Omega,\vartheta)} = \|u\vartheta^{\frac{1}{p(.)}}\|_{L^{p(x)}(\Omega)} < \infty$.
The relations between the modular $\rho_{p(.),\vartheta(.)}$ and the norm $\|.\|_{L^{p(x)}(\Omega,\vartheta)}$ are as follows:

$$\min\{\rho_{p(.),\vartheta(.)}(u)^{\frac{1}{p^-}}, \rho_{p(.),\vartheta(.)}(u)^{\frac{1}{p^+}}\} \leq \|u\|_{L^{p(x)}(\Omega,\vartheta)} \leq \max\{\rho_{p(.),\vartheta(.)}(u)^{\frac{1}{p^-}}, \rho_{p(.),\vartheta(.)}(u)^{\frac{1}{p^+}}\} \tag{2.1}$$

and

$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega,\vartheta)}, \|u\|^{p^+}_{L^{p(x)}(\Omega,\vartheta)}\} \leq \rho_{p(.),\vartheta(.)}(u) \leq \max\{\|u\|^{p^-}_{L^{p(x)}(\Omega,\vartheta)}, \|u\|^{p^+}_{L^{p(x)}(\Omega,\vartheta)}\}. \tag{2.2}$$

If $0 < k \leq \vartheta$, then we have $L^{p(x)}(\Omega, \vartheta) \hookrightarrow L^{p(x)}(\Omega)$, since one easily sees that

$$k\int_\Omega |u(x)|^{p(x)} dx \leq \int_\Omega \vartheta(x)\,|u(x)|^{p(x)} dx$$

and

$$k\|u\|_{L^{p(x)}(\Omega)} \leq \|u\|_{L^{p(x)}(\Omega,\vartheta)}.$$

Furthermore, $L^{p'(x)}(\Omega, \vartheta^*)$ is the dual space of $L^{p(x)}(\Omega, \vartheta)$, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$ and $\vartheta^* = \vartheta^{1-p'(.)} = \vartheta^{\frac{-1}{1-p(.)}}$. For more details, we refer [3]-[5].

Lemma 2.1. ([2]) Let $\Omega \subset \mathbb{R}^N$ is bounded. The function $u \in C_0^\infty(\Omega)$ satisfies Poincar inequality in $L^1_m(\Omega)$ if and only if

$$\int_\Omega m(x)|u(x)|dx \leq c_1(\mathrm{diam}(\Omega) \int_\Omega m(x)|\nabla u(x)|dx$$

is holds, where m is a weight function.

We define $W_{p(.)}(\Omega) = \{\vartheta \in L^1_{Loc}(\Omega) / \quad \vartheta^{\frac{-1}{p(.)-1}} \in L^1_{Loc}(\Omega)\}$.

Proposition 2.2 ([4])**.** If $\vartheta \in W_{p(.)}(\Omega)$, then

$$L^{p(x)}(\Omega, \vartheta) \hookrightarrow L^1_{Loc}(\Omega) \hookrightarrow D'(\Omega),$$

where $D'(\Omega)$ is distribution space.

We put the weighted variable Sobolev spaces $W^{k,p(x)}(\Omega, \vartheta)$ by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega, \vartheta): \quad D^\alpha u \in L^{p(x)}(\Omega, \vartheta), \quad |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}},$$

is the derivation in distribution sense, with $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$.
The space $W^{k,p(x)}(\Omega, \vartheta)$, equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega,\vartheta)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^{p(x)}(\Omega,\vartheta)},$$

also becomes a Banach, separable and reflexive space. In particular, the space $W^{1,p(x)}(\Omega, \vartheta)$ is defined by

$$W^{1,p(.)}(\Omega, \vartheta) = \left\{u \in L^{p(.)}(\Omega, \vartheta) \quad / \quad |\nabla u| \in L^{p(x)}(\Omega, \vartheta)\right\}$$

to be the weighted Sobolev space with the norm

$$\|u\|_{W^{1,p(.)}(\Omega,\vartheta)} = \|u\|_{L^{p(x)}(\Omega,\vartheta)} + \|\nabla u\|_{L^{p(x)}(\Omega,\vartheta)}$$

Also define $W_0^{1,p(x)}(\Omega, \vartheta) \subset W^{1,p(x)}(\Omega, \vartheta)$ to be a closure of the set $C_0^\infty(\Omega)$ (smooth functions with compact support in Ω) with respect to the norm $\|.\|_{W^{1,p(.)}(\Omega,\vartheta)}$.
In the same way as in [9] Theorem 2.1, we show the following proposition

Proposition 2.3. Assume that the boundary of Ω possesses the cone property and $p \in C_+(\overline{\Omega})$. Suppose that $c \in L^{\beta(x)}(\Omega)$, $c(x) > 0$ for a.e. $x \in \Omega$. If $q(x) \in C_+(\overline{\Omega})$ and

$$q(x) < \frac{\beta(x)-1}{\beta(x)} p^*(x), \quad \forall x \in \overline{\Omega}.$$

Then

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega, c(x))$$

is a compact embedding.

Proposition 2.4. ([10]) Let $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$. Then, the following embedding is continuous

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Remark 2.5. If we change $\leq$ by $<$, the embedding is compact.

Furthermore, we define $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,$$

, then the following relations hold

Proposition 2.6 ([6, 7, 8])**.** If $u, u_n \in L^{p(x)}(\Omega)$, $(n = 1, 2, ...)$ then the following statements are equivalent:

(i) $\lim_{n \longrightarrow \infty} |u_n - u|_{p(x)} = 0$;

(ii) $\lim_{n \longrightarrow \infty} \rho_{p(x)}(u_n - u) = 0$;

(iii) $u_n \longrightarrow u$ in measure in Ω and $\lim_{n \longrightarrow \infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

Proposition 2.7 ([6],[7],[8])**.** If $u, u_n \in L^{p(x)}(\Omega)$, $(n = 1, 2, ...)$, we have

(1) $|u|_{p(x)} < 1$ (respectively=1; > 1) $\Longleftrightarrow \rho_{p(x)}(u) < 1$ (respectively $= 1$; > 1);

(2) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;

(3) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;

(4) $|u_n|_{p(x)} \to 0$ (respectively $\to \infty$) $\Longleftrightarrow \rho_{p(x)}(u_n) \to 0$ (respectively $\to \infty$).

Remark 2.8. If we consider $\rho_{p(.),\vartheta(.)}(u) = \int_\Omega \vartheta(x)|u|^{p(x)} dx$, instead of $\rho_{p(x)}(u)$, then the statements of Proposition 2.6 and Proposition 2.7 also hold for $u, u_n \in W^{1,p(x)}(\Omega, \vartheta(x))$.

Lemma 2.9. (See [11]) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous function such that $(F(x), x) \geq 0$, for all x verifying $|x| = R > 0$. Then there exists $\gamma \in B_R(0)$ such that $F(\gamma) = 0$.

3 Main result

Before we consider the existence of weak solution of problem 1.1, we present and study the weighted variable exponent Sobolev spaces.
For $\alpha \in \mathbb{R}^+$, we define the following the weighted variable exponent space

$$L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) := \left\{u = u(x),\ x \in \Omega,\ \int \frac{1}{d^\alpha} |u|^{p(x)} < \infty\right\}$$

equipped with the norm

$$|u|_{\alpha,p(x)} := \inf\left\{\mu > 0/ \ \int_\Omega \frac{1}{d^\alpha} \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}, \tag{3.1}$$

and

$$W^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) := \left\{u,\ \int_\Omega \frac{1}{d^\alpha} |\nabla u|^{p(x)} + |u|^{p(x)}_{\alpha,p(x)} < \infty\right\} \tag{3.2}$$

equipped with the norm

$$\|u\|_X = |u|_{1,\alpha,p(x)} := |\nabla u|_{\alpha,p(x)} + |u|_{\alpha,p(x)}. \tag{3.3}$$

Lemma 3.1. Let $\alpha \in \mathbb{R}^+$. The space $W^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$, equipped with the norm $|.|_{1,\alpha,p(x)}$, is Banach space.

Proof. Let $(u_n)_n$ be a sequence in X_γ. Suppose that $(u_n)_n$ is Cauchy sequence in X_γ. It is easy to see that $(u_n)_n$ $(resp. \left((\partial u_n/\partial x_1, .., \partial u_n/\partial x_N)\right)_n)$ is also Cauchy sequence in $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ $\left(resp. \left(L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)\right)^N\right)$. Using the fact that the space $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ is Banach space, we obtain that there exist u and w_i, $i = 1, 2, .., N$, such that the sequence $(u_n)_n$ converges to u in $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ and the sequence $(\partial u_n/\partial x_i)$ converges to w_i in $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ for $i = 1, .., N$. For every $\psi \in D(\Omega)$, we have

$$\left|\int_\Omega u_n\psi \mathrm{dx} - \int_\Omega u\psi \mathrm{dx}\right| \leq \int_\Omega |u_n - u||\psi| \mathrm{dx} \tag{3.4}$$

and

$$\begin{aligned}\int_\Omega |u_n - u||\psi|\mathrm{dx} &= \int_\Omega \frac{1}{d^{\alpha p(x)}}|u_n - u|\frac{1}{d^{-\alpha p(x)}}|\psi|\mathrm{dx} \\ &\leq |u_n - u|_{\alpha,p(x)} \left|\left|\frac{1}{d^{-\alpha p(x)}}\psi\right|\right|_{p'(x)} \\ &\leq |u_n - u|_{\alpha,p(x)} \|\psi\|_\infty \left|\left|\frac{1}{d^{-\alpha p(x)}}\right|\right|_{L^{p'(x)}(supp\psi)}\end{aligned}$$

where $supp\psi \subset \Omega$ denotes the support of ψ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. We deduce

$$\left|\int_\Omega u_n\psi \mathrm{dx} - \int_\Omega u\psi \mathrm{dx}\right| \leq |u_n - u|_{\alpha,p(x)} \|\psi\|_\infty \left|\left|\frac{1}{d^{-\alpha p(x)}}\right|\right|_{L^{p'(x)}(supp\psi)} \tag{3.5}$$

Since $\left|\left|\frac{1}{d^{-\alpha p(x)}}\right|\right|_{L^{p'(x)}(supp\varphi)} < \infty$ and $u_n \longrightarrow u$ in $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ we get

$$\int_\Omega u_n\psi \mathrm{dx} \longrightarrow \int_\Omega u\psi \mathrm{dx} \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.6}$$

Similarly, by $\partial u_n/\partial x_i \longrightarrow w_i$ in $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$, we obtain

$$\int_\Omega \frac{\partial u_n}{\partial x_i}\psi \mathrm{dx} \longrightarrow \int_\Omega w_i\psi \mathrm{dx},\ i = 1,..,N \tag{3.7}$$

as $n \longrightarrow +\infty$. Then,

$$\int_\Omega w_i\psi \mathrm{dx} = \lim_{n\longrightarrow\infty}\int_\Omega \frac{\partial u_n}{\partial x_i}\psi \mathrm{dx} \tag{3.8}$$

$$= -\lim_{n\longrightarrow\infty}\int_\Omega u_n\frac{\partial \psi}{\partial x_i}\mathrm{dx} \tag{3.9}$$

$$= -\int_\Omega u\frac{\partial \psi}{\partial x_i}\mathrm{dx} \tag{3.10}$$

$$= \int_\Omega \frac{\partial u}{\partial x_i}\psi \mathrm{dx},\ i = 1,...,N. \tag{3.11}$$

We deduce that $\nabla u = w$ and $u_n \longrightarrow u$ in $W^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$. This completes the proof. Q.E.D.

Lemma 3.2. Let $\alpha \in \mathbb{R}^+$. The space $X_\alpha := W^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ equipped with the norm $|.|_{1,p(x),\alpha}$ is reflexive.

Proof. It is easy to verify that $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ is uniformly convex. Therefore, $L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ is reflexive. Consider the functional

$$T:\ X_\alpha \to E := L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \times L^{p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)^N$$
$$u \to T(u) = (u, \nabla u)$$

Since T is an isometry, we obtain $T(X_\alpha)$ is a closed subspace of E. Then $T(X_\gamma)$ is reflexive, therefore X_α is reflexive. Q.E.D.

Lemma 3.3. Assume that $p, q \in L^\infty_+(\Omega)$. If $q(x) \leq p(x)$. Then the embedding

$$L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right) \hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$$

is continuously.

Lemma 3.4. Assume that $p, q \in L^\infty_+(\Omega)$. If $q(x) \leq p(x)$. Then the embedding

$$W^{m,p(x)}\left(\Omega, \frac{1}{d^\gamma}\right) \hookrightarrow W^{m,q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$$

is continuously.

Proof. The proof of Lemma 3.3 and 3.4 are semitric with the proof of Lemma 3.5. Q.E.D.

The following lemma concerns a result of compact embedding.

Lemma 3.5. Let $p, q \in L^{\infty}_{+}(\Omega)$, we assume $mp(x) \leq n$ and $q(x) < \frac{np(x)}{n-mp(x)}$ $\forall x \in \overline{\Omega}$. Then

$$X_\gamma =: W^{m,p(x)}\left(\Omega, \frac{1}{d^\gamma}\right) \hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$$

is continuous and compact embedding. We write $X_\gamma \hookrightarrow\hookrightarrow L^{q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$.

Proof. For positive constant r with $mr < n$, denote $r^* = \frac{nr}{n-mr}$.
Under the asumptions it is easy to see that for arbitrary $x \in \overline{\Omega}$, we can find a neighborhood U_x in $\overline{\Omega}$ such that $q^+(U_x) < (p^-(U_x))^*$, where

$$p^-(U_x) = \inf\{p(y)/ \quad y \in U_x\}; \quad q^+(U_x) = \sup\{q(y)/ \quad y \in U_x\}.$$

Now $\{U_x\}_{x\in\overline{\Omega}}$ is an open covering $\{U_i \quad / \quad i = 1, 2, ..., s\}$ and denoting

$$p_i^- = p^-(U_i) \qquad q_i^+ = q^+(U_i),$$

it is obvious that if $u \in W^{m,p(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$ then $u \in W^{m,p(x)}\left(U_i, \frac{1}{d^\gamma}\right)$, and thus from the lemma 3.4, $u \in W^{m,p^-}\left(U_i, \frac{1}{d^\gamma}\right)$. Therefore by the well-known Sobolev imbedding theorem we have continuous and compact imbedding,

$$W^{m,p_i^-}\left(U_i, \frac{1}{d^\gamma}\right) \hookrightarrow L^{q_i^+}\left(U_i, \frac{1}{d^\gamma}\right),$$

so for every U_i, $i = 1, 2, ..., s$ (see [1]), we have $u \in L^{q(x)}\left(U_i, \frac{1}{d^\gamma}\right)$ and therefore $u \in L^{q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$. We can now assert that $W^{m,p(x)}\left(\Omega, \frac{1}{d^\gamma}\right) \subset L^{q(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$ and the imbedding is continuous and compact.

Q.E.D.

Lemma 3.6. Let $0 \leq \gamma < \alpha + p^-$. Then

$$W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right) \hookrightarrow L^{p*(x)}(\Omega)$$

Proof. Let $\sigma > 0$ small enough we define $\Omega_\sigma := \{x \in \Omega \quad / \quad d(x) > \sigma\}$. Consider they maps:

$$I_\sigma : W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right),$$

$$I_\sigma^1 : W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \longrightarrow W_0^{1,p(x)}(\Omega_\sigma),$$

$$I_\sigma^2 : W_0^{1,p(x)}(\Omega_\sigma) \longrightarrow L^{p(x)}(\Omega_\sigma)$$

and

$$I_\sigma^3 : L^{p(x)}(\Omega_\sigma) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right).$$

Here $I_\sigma^1 u(x) = u(x)$, $x \in \Omega_\sigma$ is the " restriction " (bounded and linear map) from $W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ into $W_0^{1,p(x)}(\Omega_\sigma)$; $I_\sigma^2 u(x) = u(x)$, $x \in \Omega_\sigma$ is " embedding " (compact linear map) from $W_0^{1,p(x)}(\Omega_\sigma)$ into $L^{p(x)}(\Omega_\sigma)$; $I_\sigma^3 u(x) = u(x)$, $x \in \Omega_\sigma$ and $I_\sigma^3 u(x) = 0$, $x \in \Omega \backslash \Omega_\sigma$ is " zero extension " (bounded and linear map) from $L^{p(x)}(\Omega_\sigma)$ into $L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$.
Then for any $\sigma > 0$, $I_\sigma : W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \longrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$ is a compact linear map. Denote by

$$I : W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right) \hookrightarrow L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right),$$

the embedding of $W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ into $L^{p(x)}\left(\Omega, \frac{1}{d^\gamma}\right)$. Then for $u \in W_0^{1,p(x)}\left(\Omega, \frac{1}{d^\alpha}\right)$ there exists $K > 0$ such that

$$\begin{aligned} |(I_\sigma - I)(u)|_{\gamma,p(x)} &= \int_{\Omega\backslash\Omega_\sigma} \frac{1}{d^\gamma(x)} |u(x)|^{p(x)} dx = \int_{\Omega\backslash\Omega_\sigma} \frac{1}{d^\gamma} d^{\alpha+p(x)} \frac{|u(x)|^{p(x)}}{d^{\alpha+p(x)}} dx \\ &\leq \max_{x\in\Omega\backslash\Omega_\sigma} d^{\alpha+p^+-\gamma} \int_{\Omega\backslash\Omega_\sigma} \frac{1}{d^{\alpha+p^-}} |u(x)|^{p(x)} dx \\ &\leq \max_{x\in\Omega\backslash\Omega_\sigma} d^{\alpha+p^+-\gamma} |u|_{p(x),\alpha+p^-}^{p^-} \\ &\leq \max_{x\in\Omega\backslash\Omega_\sigma} d^{\alpha+p^+-\gamma} |u|_{p(x),\alpha}^{p^-} \\ &\leq K \max_{x\in\Omega\backslash\Omega_\sigma} d^{\alpha+p^+-\gamma} \|u\|_X^{p^-}. \end{aligned}$$

The assertion now follows from the fact that $\max_{x\in\Omega\backslash\Omega_\sigma} d^{\alpha+p^+-\gamma} \longrightarrow 0$ as $\sigma \longrightarrow 0$.

Q.E.D.

3.1 Proof of the main result

We recall that $u \in X_\alpha$ is a weak solution to the problem (1.1) if it verifies

$$M\left(\int_\Omega \frac{1}{p(x)} \frac{1}{d^\alpha} |\nabla u|^{p(x)}\right) \int_\Omega \frac{1}{d^\alpha} |\nabla u|^{p(x)-2} \nabla u \nabla v \mathrm{dx} - \int_\Omega \frac{1}{d^\beta} h(x,u).v \mathrm{dx} - \int_\Omega \frac{1}{d^\gamma} g(x,\nabla u).v \mathrm{dx} = 0 \quad \forall v \in X$$

Proof of ***Theorem 1.1*** . Let $\kappa = \{e_1, e_2, ..., e_n, ...\} \subset X$ such that

$$E = \overline{\operatorname{span}\{e_1, e_2, ..., e_n\}}$$

Define $\vartheta_n = \{e_1, e_2, ..., e_n\}$. It is known that ϑ_n and $\mathbb{R}^n$ are isomorphic and for $\xi \in \mathbb{R}^n$, we have an unique $v \in \vartheta_n$ by the identification $\varphi : \xi \longrightarrow \sum_{i=1}^n \xi_i e_i = v$.
Denote the function $F = (F_1, F_2, ..., F_n) : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$F_i(u) = M\left(\int_\Omega \frac{1}{p(x)} \frac{1}{d^\alpha} |\nabla u|^{p(x)}\right) \int_\Omega \frac{1}{d^\alpha} |\nabla u|^{p(x)-2} \nabla u \nabla e_i \mathrm{dx} - \int_\Omega (g(x,\nabla u) + h(x,u))\, e_i \mathrm{dx} \qquad u \in \vartheta_i.$$

We show the existence of weak solutions $u_n \in \vartheta_n$ for the problem

$$\begin{aligned} M\left(\int_\Omega \frac{1}{p(x)}\frac{1}{d^\alpha}|\nabla u_n|^{p(x)}\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p(x)-2}\nabla u_n \nabla e_i \mathrm{dx} &= \int_\Omega \frac{1}{d^\beta}h(x,u_n).e_i\mathrm{dx} \\ &+ \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n).e_i\mathrm{dx} \\ & \quad u_n \in \vartheta_i \end{aligned}$$

For $u_n \in \vartheta_n$, we have that

$$\begin{aligned} (F(u_n),u_n) &= M\left(\int_\Omega \frac{1}{p(x)}\frac{1}{d^\alpha}|\nabla u_n|^{p(x)}\right)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p(x)}\mathrm{dx} \\ &\quad -\int_\Omega \frac{1}{d^\beta}h(x,u_n)u_n\mathrm{dx} - \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n)u_n\mathrm{dx} \\ &\geq m_0\rho_{p(x),\alpha}(u_n) - \int_\Omega \frac{1}{d^\beta}h(x,u_n).u_n\mathrm{dx} - \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n).u_n\mathrm{dx} \\ &\geq m_0 k_0\|u_n\|_X^{p^0} - \int_\Omega \frac{1}{d^\beta}h(x,u_n).u_n\mathrm{dx} - \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n).u_n\mathrm{dx}. \end{aligned}$$

where

$$p^0 = \begin{cases} p^- & \text{if } \|u\|_X < 1 \\ p^+ & \text{if } \|u\|_X > 1. \end{cases} \tag{3.12}$$

According to condition (H_2) and (H_3) we have

$$\begin{aligned} \int_\Omega \frac{1}{d^\beta}h(x,u_n)u_n\mathrm{dx} &\leq \|a_1\|_{p'(x);\beta}\rho_{p(x),\beta}(u_n)^{\frac{1}{p^0}} + \|b_1\|_{\frac{p(x)}{p(x)-(r_1(x)+1)};\beta}\rho_{p(x),\beta}(u_n)^{\frac{r_1^+ +1}{p^0}} \\ &\leq k_1\|a_1\|_{p'(x);\beta}\|u_n\|_X + k_2\|b_1\|_{\frac{p(x)}{p(x)-(r_1(x)+1)};\beta}\|u_n\|_X^{r_1^+ +1}. \end{aligned}$$

And

$$\begin{aligned} \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n)u_n\mathrm{dx} &\leq \|a_2\|_{p'(x);\gamma}\rho_{p(x),\gamma}(u_n)^{\frac{1}{p^0}} + \||b_2|^{p'(x)}\|_{\frac{p(x)}{p(x)-r_2(x)};\gamma}\rho_{p(x),\gamma}(u_n)^{\frac{r_2^+ +1}{p^0}} \\ &\leq k_3\|a_2\|_{p'(x);\gamma}\|u_n\|_X + k_4\||b_2|^{p'(x)}\|_{\frac{p(x)}{p(x)-r_2(x)};\gamma}\|u_n\|_X^{r_2^+ +1}. \end{aligned}$$

Then

$$\begin{aligned} (F(u_n),u_n) &\geq m_0k_0\|u_n\|_X^{p^0} - k_1\|a_1\|_{p'(x);\beta}\|u_n\|_X - k_2\|b_1\|_{\frac{p(x)}{p(x)-(r_1(x)+1)};\beta}\|u_n\|_X^{r_1^+ +1} \\ &\quad -k_3\|a_2\|_{p'(x);\gamma}\|u_n\|_X - k_4\||b_2|^{p'(x)}\|_{\frac{p(x)}{p(x)-r_2(x)};\gamma}\|u_n\|_X^{r_2^+ +1}. \end{aligned}$$

Since $r_i^+ + 1 < p^0$, $i = 1,2$, there exist positive numbers ρ and R such that

$$(F(u_n),u_n) \geq \rho > 0 \qquad \text{on} \quad \|u\|_X = R.$$

F is continuous, by lemma 2.9, the equation (3.1) has a solution u_n in $\vartheta_n \subset X$ with $\|u_n\|_X \leq R$. we may assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \qquad \text{in} \quad X,$$

$$u_n \longrightarrow u \qquad \text{a.e.} \quad x \in \Omega.$$

By using the Dominated Convergence Theorem, we get

$$\int_\Omega h(x,u_n)w\mathrm{dx} \longrightarrow \int_\Omega h(x,u)w\mathrm{dx} \qquad \text{for} \quad w \in \vartheta_n$$

In view of condition (H_3), we have

$$\|g(x,\nabla u_n)\|_{\frac{p(x)}{r_2(x)},\gamma} \leq \left(\int_\Omega \frac{1}{d^\gamma}\left|a_2 + b_2|\nabla u_n|^{r_2(x)}\right|^{\frac{p(x)}{r_2(x)}}\mathrm{dx}\right)^{\frac{r_2(x)}{p(x)}}$$

$$\Rightarrow \|g(x,\nabla u_n)\|_{\frac{p(x)}{r_2(x)},\gamma} \leq \left(2^{\frac{p^0}{r_2^0}-1}\left(\int_\Omega \frac{1}{d^\gamma}|a_2|^{\frac{p(x)}{r_2(x)}}\mathrm{dx} + \int_\Omega \frac{1}{d^\gamma}|b_2|^{\frac{p(x)}{r_2(x)}}|\nabla u_n|^{p(x)}\mathrm{dx}\right)\right)^{\frac{r_2(x)}{p(x)}}$$

$$\Rightarrow \|g(x,\nabla u_n)\|_{\frac{p(x)}{r_2(x)},\gamma} \leq \left(\|a_2\|^{\frac{p^0}{r_2^0}}_{\frac{p(x)}{r_2(x)},\gamma} + k_5||b_2|^{\frac{p^0}{r_2^0}}|_\infty.\|u\|_X^{r_2(x)}\right)^{\frac{r_2(x)}{p(x)}}$$

implique

$$\|g(x,\nabla u_n)\|_{\frac{p(x)}{r_2(x)},\gamma} \leq \left(\|a_2\|_{\frac{p(x)}{r_2(x)},\gamma} + k_5||b_2|^{\frac{p^*}{r_2^*}}|_\infty.R^{r_2^+}\right)^{\frac{r_2^+}{p^*}} < \infty. \tag{3.13}$$

From the reflexivity of $L^{\frac{r_2(x)}{p(x)}}\left(\Omega,\frac{1}{d^\gamma}\right)$, passing to a subsequence if necessary; there is $g(x,\nabla u) \in L^{\frac{r_2(x)}{p(x)}}\left(\Omega,\frac{1}{d^\gamma}\right)$ such that

$$\int_\Omega \frac{1}{d^\gamma}.g(x,\nabla u_n).\varphi\mathrm{dx} \longrightarrow \int_\Omega \frac{1}{d^\gamma}.g(x,\nabla u).\varphi\mathrm{dx} \quad \forall\varphi \in L^{q(x)}\left(\Omega,\frac{1}{d^\gamma}\right)$$

with $\frac{r_2(x)}{p(x)} + \frac{1}{q(x)} = 1$.
We have

$$M(\|u_n\|_X^p)\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\mathrm{dx} = \int_\Omega \frac{1}{d^\beta}h(x,u_n).\varphi\mathrm{dx} + \int_\Omega \frac{1}{d^\gamma}g(x,\nabla u_n).\varphi\mathrm{dx} \qquad \varphi \in X,$$

on the other hand we have $u_n \rightharpoonup u$, thus when $n \longrightarrow +\infty$, we get

$$\int_\Omega \frac{1}{d^\beta}\left(h(x,u_n) - h(x,u)\right).(u_n - u)\,\mathrm{dx} \longrightarrow 0 \tag{3.14}$$

thus

$$\int_\Omega \frac{1}{d^\alpha}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n - u)\mathrm{dx} \longrightarrow 0.$$

Then

$$\|u_n - u\|_X \longrightarrow 0 \tag{3.15}$$

by using the inequality (see [20])

$$\left[\left(|\psi|^{p(x)-2}\psi - |\xi|^{p(x)-2}\xi\right)(\psi-\xi)\right]\cdot\left(|\psi|^{p(x)} - |\xi|^{p(x)}\right)^{\frac{2-p(x)}{p(x)}} \geq (p-1)|\psi-\xi|^{p(x)}, \quad 1<p<2;$$

$$\left(|\psi|^{p(x)-2}\psi - |\xi|^{p(x)-2}\xi\right)(\psi-\xi) \geq \left(\frac{1}{2}\right)^{p(x)} |\psi-\xi|^{p(x)}, \quad p \geq 2.$$

Then

$$u_n \longrightarrow u \quad \text{in} \quad X. \tag{3.16}$$

M is continuous, this implies that

$$M\left(\int_\Omega \frac{1}{d^\alpha}|\nabla u_n|^{p(x)}\mathrm{dx}\right) \longrightarrow M\left(\int_\Omega \frac{1}{d^\alpha}|\nabla u|^{p(x)}\mathrm{dx}\right) \tag{3.17}$$

So we obtain that u is a weak solution to the problem (1.1), and from (H_1), we obtain $u \neq 0$.

Q.E.D.

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